$$\Phi_{nkl}^{c} = \int_0^{2\pi} \cos k\varphi \int_0^1 \xi_1 J_k(\nu_n^k \xi_1) \int_0^1 \Phi \sin \lambda_l (1-\xi_2)\, d\xi_2 d\xi_1 d\varphi$$

$$\psi_{kl}^{*c} = \int_0^{2\pi} \cos k\varphi \int_0^1 \psi^* \sin \lambda_l (1-\xi_2)\, d\xi_2 d\varphi \qquad (3.7)$$

$$\Omega_{kn}^{c} = \int_0^{2\pi} \cos k\varphi \int_0^1 \Omega \xi_1 J_k(\nu_n^k \xi_1)\, d\xi_1 d\varphi$$

Replacing the superscript c in (3.6) and (3.7) by s and $\cos k\varphi$ and $\sin k\varphi$ in the right hand sides of (3.7), we obtain the equation for V_{nkl}^s.

Since (3.6) and the analogous equations for V_{nkl}^s are first order linear, they can be easily integrated by quadratures. Insertion of the values of V_{nkl}^c and V_{nkl}^s thus obtained into (3.5), completes the solution of the problem.

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SOLUTION OF TWO-DIMENSIONAL DOUBLY-PERIODIC PROBLEMS OF THE THEORY OF STEADY VIBRATIONS OF VISCOELASTIC BODIES

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The first and second boundary value problems of the steady vibrations of a viscoelastic body occupying a domain in the form of a plane with an infinite number of identical circular holes which form an oblique-angled grating, are considered. The problems are reduced to infinite systems of algebraic equations with normal-type determinant. Reasoning of a physical nature is utilized in providing the uniqueness of the solutions of these systems.

An extensive literature has been devoted to periodic and doubly-periodic problems of plane static elasticity theory. A very detailed exposition of the results obtained is contained in the survey [1].

Let us place the origin 0_{qs} of a r_{qs}, θ_{qs} polar coordinate system at the center of each of the holes, where r_{qs} is a dimensionless coordinate expressed in fractions of the hole radius R.

Let us introduce the following notation: Γ_{qs} is the contour of the qsth hole; R_{qs}^{00}, θ_{qs}^{00} the polar coordinates of the pole 0_{00} in the qsth coordinate system; $U(\theta_{qs})e^{-i\omega t}$, $V(\theta_{qs})e^{-i\omega t}$ displacement components given on Γ_{qs} (the second boundary value problem); $P(\theta_{qs})e^{-i\omega t}$, $T(\theta_{qs})e^{-i\omega t}$ the normal and tangential components of the external forces applied to the contour Γ_{qs} (first boundary value problem).

As is known, two-dimensional problems of the steady vibrations of a linear viscoelastic solid reduce to the solution of two equations

$$\Delta\Phi + \alpha^2\Phi = 0, \qquad \Delta\Psi + \beta^2\Psi = 0 \tag{1}$$

where α^2 and β^2 are some complex numbers with positive imaginary parts. Their expressions in terms of the mechanical parameters of the body are defined by the customary laws of viscoelasticity.

The boundary conditions on each of the contours Γ_{qs} can be represented in the form

$$\begin{aligned} a\alpha^2\Phi + \frac{1}{r_{qs}}\left(\frac{\partial\Phi}{\partial r_{qs}} + \frac{1}{r_{qs}}\frac{\partial^2\Phi}{\partial\theta_{qs}^2} + \frac{1}{r_{qs}}\frac{\partial\Psi}{\partial\theta_{qs}} - \frac{\partial^2\Psi}{\partial\theta_{qs}\partial r_{qs}}\right)\Bigg|_{r_{qs}=1} &= F_1(\theta_{qs}) \\ b\beta^2\Psi + \frac{1}{r_{qs}}\left(\frac{\partial\Psi}{\partial r_{qs}} + \frac{1}{r_{qs}}\frac{\partial^2\Psi}{\partial\theta_{qs}^2} - \frac{1}{r_{qs}}\frac{\partial\Phi}{\partial\theta_{qs}} + \frac{\partial^2\Phi}{\partial\theta_{qs}\partial r_{qs}}\right)\Bigg|_{r_{qs}=1} &= F_2(\theta_{qs}) \end{aligned} \tag{2}$$

where $a = \text{const} \neq 0, b = \text{const} \neq 0$ for the first boundary value problem

$$F_1(\theta_{qs}) = -\frac{R^2}{2\mu^*}P(\theta_{qs}) \text{ and } F_2(\theta_{qs}) = \frac{R^2}{2\mu^*}T(\theta_{qs})$$

and $a = b = 0$ for the second boundary value problem

$$F_1(\theta_{qs}) = R\left[U(\theta_{qs}) + \frac{\partial V(\theta_{qs})}{\partial\theta_{qs}}\right], \qquad F_2(\theta_{qs}) = R\left[\frac{\partial U(\theta_{qs})}{\partial\theta_{qs}} - V(\theta_{qs})\right]$$

Utilizing the method the authors used earlier in solving problems of elastic vibrations in the case of multiconnected domains of finite connectivity [2], we write the solution of (1) as

$$\Phi = \sum_n\sum_q\sum_s A_n H_n(\alpha r_{qs})\, e^{in\theta_{qs}}, \qquad \Psi = \sum_n\sum_q\sum_s B_n H_n(\beta r_{qs})\, e^{in\theta_{qs}} \tag{3}$$

where H_n is the Hankel function of the first kind (*)

The constants A_n and B_n can be determined from the boundary conditions on any of the contours Γ_{qs}, for example on Γ_{00}. To do this it is necessary to represent the potentials Φ and Ψ in the coordinates r_{00} and θ_{00}, which can be realized by utilizing the following fromulas:

$$H_n(cr_{qs})\, e^{in\theta_{qs}} = \sum_p H_{n-p}(cR_{qs}^{00})\, e^{i(n-p)\theta_{qs}^{00}} J_p(cr_{00})\, e^{ip\theta_{00}} \qquad (r_{00} < R_{qs}^{00}) \tag{4}$$

Here J_p is a Bessel function of the first kind. After some manipulation we have

$$\begin{aligned} \Phi(r_{00}, \theta_{00}) &= \sum_n [A_n H_n(\alpha r_{00}) + S_n J_n(\alpha r_{00})]\, e^{in\theta_{00}} \\ \Psi(r_{00}, \theta_{00}) &= \sum_n [B_n H_n(\beta r_{00}) + Q_n J_n(\beta r_{00})]\, e^{in\theta_{00}} \end{aligned} \tag{5}$$

where

$$\begin{aligned} S_n &= \sum_p \sum_q{}' \sum_s{}' A_p H_{p-n}(\alpha R_{qs}^{00}) \exp i(p-n)\,\theta_{qs}^{00} \\ Q_n &= \sum_p \sum_q{}' \sum_s{}' B_p H_{p-n}(\beta R_{qs}^{00}) \exp i(p-n)\,\theta_{qs}^{00} \end{aligned} \tag{6}$$

The primes on the sums in (6) mean that components for which $q = s = 0$ have been discarded therein.

*) Here and in the sums below the summation indices vary between $-\infty$ and $+\infty$.

Let us expand the right sides of boundary conditions (2) in the Fourier series

$$F_l(\theta_{00}) = \sum_n f_{l,n} e^{in\theta_{00}} \qquad (l = 1, 2) \tag{7}$$

Substituting (5) and (7) into (2) results in an infinite system of algebraic equations with the unknowns A_n and B_n

$$\begin{gathered} A_n\bar{u}_{n,1}(\alpha) + B_n\bar{u}_{n,2}(\beta) = f_{1,n} - S_n u_{n,1}(\alpha) - Q_n u_{n,2}(\beta) \\ -A_n\bar{u}_{n,2}(\alpha) + B_n\bar{u}_{n,1}(\beta) = f_{2,n} + S_n u_{n,2}(\alpha) - Q_n u_{n,1}(\beta) \qquad (n = 0, \pm 1, \pm 2, \ldots) \end{gathered} \tag{8}$$

wherein we have used the notation

$$u_{n,1}(x) = (cx^2 - n^2) J_n(x) + xJ_n'(x), \; u_{n,2}(x) = in\,[(J_n(x) - xJ_n'(x)]$$

$$c = a \;\text{ for }\; x = \alpha, \; c = b \;\text{ for }\; x = \beta$$

Formulas for $\bar{u}_{n,1}$ and $\bar{u}_{n,2}$ are obtained from $u_{n,1}$ and $u_{n,2}$ by replacing $J_n(x)$ and $J'_n(x)$ by $H_n(x)$ and $H_n'(x)$.

Let us now make a change of unknowns A_n and B_n by assuming

$$A_n\bar{u}_{n,1}(\alpha) + B_n\bar{u}_{n,2}(\beta) = C_n$$

$$-A_n\bar{u}_{n,2}(\alpha) + B_n\bar{u}_{n,1}(\beta) = D_n$$

whereupon the system (8) takes the canonical form

$$\begin{gathered} C_n = \sum_p [\zeta_{n,p}(\alpha, \beta) C_p + \eta_{n,p}(\alpha, \beta) D_p] + f_{1,n} \\ D_n = \sum_p [-\eta_{n,p}(\beta, \alpha) C_p + \zeta_{n,p}(\beta, \alpha) D_p] + f_{2,n} \quad (n = 0, \pm 1, \pm 2, \ldots) \end{gathered} \tag{9}$$

The following notation has been introduced for the coefficients in the system (9)

$$\begin{gathered} \begin{matrix} \zeta_{n,p}(\alpha, \beta) \\ \eta_{n,p}(\alpha, \beta) \end{matrix} = \frac{1}{\Delta_p}\left[\begin{matrix} -\bar{u}_{p,1}(\beta) \\ \bar{u}_{p,2}(\beta) \end{matrix} u_{n,1}(\alpha) \sum_q{}' \sum_s{}' H_{p-n}(\alpha R_{qs}^{00}) e^{i(p-n)\theta_{qs}^{00}} - \right. \\ \left. - \begin{matrix} \bar{u}_{p,2}(\alpha) \\ \bar{u}_{p,1}(\alpha) \end{matrix} u_{n,2}(\beta) \sum_q{}' \sum_s{}' H_{p-n}(\beta R_{qs}^{00}) e^{i(p-n)\theta_{qs}^{00}} \right] \\ \Delta_p = \bar{u}_{p,1}(\alpha)\bar{u}_{p,1}(\beta) + \bar{u}_{p,2}(\alpha)\bar{u}_{p,2}(\beta) \end{gathered} \tag{10}$$

Let us show that the determinant of the infinite system (9) is a normal-type determinant [3]. To do this it is evidently sufficient to prove the convergence of the double series

$$\sum_n \sum_p |\zeta_{n,p}(\alpha, \beta)| \tag{11}$$

To establish the convergence of the series (11) we need the upper bound of the absolute sum

$$\left| \sum_q{}' \sum_s{}' H_{p-n}\left(\frac{x}{\delta} R_{qs}^{00}\right) \exp i\,(p-n)\,\theta_{qs}^{00} \right| \qquad (x = \alpha\delta, \beta\delta) \tag{12}$$

which can be obtained by utilizing the following formulas from the theory of cylinder functions:

$$\begin{gathered} H_{\nu+m}(z) = H_\nu(z) R_{m,\nu}(z) - H_{\nu-1}(z) R_{m-1,\nu+1}(z) \\ R_{m,\nu}(z) = \sum_{l=0}^{\leqslant 1/2 m} \frac{(-1)^l (m-l)!\,\Gamma(\nu+m-l)}{l!\,(m-2l)!\,\Gamma(\nu+l)} \left(\frac{z}{2}\right)^{-m+2l} \end{gathered} \tag{13}$$

Here $R_{m,\nu}(z)$ are Lommel polynomials [4], and δ is the least distance between two adjacent holes. Hence, assuming

$$\nu = 0, \; m = |p - n|, \; z = \frac{x}{\delta} R_{qs}^{00}$$

we have after some manipulations (14)

$$\Big|\sum_q{}'\sum_s{}' H_{p-n}\Big(\frac{x}{\delta}R_{qs}^{00}\Big)\exp i(p-n)\,\theta_{qs}^{00}\Big| < M\,[|R_{|p-n|,0}(i|x|)| + |R_{|p-n|-1,1}(i|x|)|]$$

where M is the greatest of the numbers

$$\Big|\sum_q{}'\sum_s{}' H_0\Big(\frac{x}{\delta}R_{qs}^{00}\Big)\exp i(p-n)\,\theta_{qs}^{00}\Big(\frac{R_{qs}^{00}}{\delta}\Big)^{-|p-n|+2l}\Big| \quad (2l \leqslant |p-n|)$$

$$\Big|\sum_q{}'\sum_s{}' H_1\Big(\frac{x}{\delta}R_{qs}^{00}\Big)\exp i(p-n)\,\theta_{qs}^{00}\Big(\frac{R_{qs}^{00}}{\delta}\Big)^{-|p-n|+2l+1}\Big| \quad (2l \leqslant |p-n|-1)$$

Proceeding from the Hurwitz limit relaionship [4]

$$\lim_{m\to\infty}\frac{\left(\frac{1}{2}z\right)^{\nu+m} R_{m,\nu+1}(z)}{\Gamma(\nu+m+1)} = J_\nu(z)$$

let us write the inequality (14) for large values of $|p-n|$ (15)

$$\Big|\sum_q{}'\sum_s{}' H_{p-n}\Big(\frac{x}{\delta}R_{qs}^{00}\Big)\exp i(p-n)\,\theta_{qs}^{00}\Big| < M\Big(\frac{2}{|x|}\Big)^{|p-n|-1}(|p-n|-1)!\,[|J_1(i|x|)| + J_0(i|x|)]$$

Taking account of asymptotic formulas for cylinder functions with large index, as well as the inequality (15), estimates for the members of the series (11) can be found from the expressions (10) for large ($|p| + |n|$). It follows from these estimates that the series (11) is majorized by the convergent series

$$\sum_n\sum_p \frac{(|n|+1)^2(|p|+1)^2(|p|+|n|)!}{|n|!\,|p|!}\Big(\frac{1}{\delta}\Big)^{|n|+|p|}$$

from which we draw a conclusion on the convergence of the former.

Therefore, the double series composed of absolute values of the coefficients of the infinite system (9) also converges, and this means that the given system has a normal-type determinant. Since the free members of the system (9) are bounded, then theorems analogous to the Kramer theorems for finite systems [3] are valid therein. The system (9) has a unique bounded solution if its determinant is nonzero. If this determinant is zero, then the corresponding homogeneous system admits of a nontrivial solution. However, this latter cannot occur since undamped free vibrations cannot exist in the medium under consideration.

Transformation of the sums into series (3) was admitted in the derivation of the system (9), as was also their term by term differentiation. These operations are legitimate if the Fourier coefficients of the right sides of the boundary conditions (2) have orders no lower than $O(n^{-4})$.

Periodic problems of steady vibrations of a viscoelastic body are obtained as a particular case from those considered above.

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